$$\mu_1 = \mu_0\left[1 - \frac{\gamma M\omega_0}{\omega_0^2 - \omega^2}\right] \tag{1}$$

$$\mu_2 = \frac{\mu_0\gamma M\omega}{\omega_0^2 - \omega^2} \tag{2}$$

where

$\omega_0 = \gamma H_1$
H_1 = internal static magnetic field
μ_0 = permeability of free space
γ = gyromagnetic ratio
M = magnetic polarization density
ω = angular frequency.

The anisotropy, μ_2/μ_1, is related to the internal field by

$$\frac{\mu_2}{\mu_1} = \frac{\gamma M\omega}{\omega_0^2 - \gamma M\omega_0 - \omega^2}. \tag{3}$$

Assuming a demagnetization factor N, the internal field is related to the external field. Expressing the magnetic polarization density in terms of a magnetic susceptibility K, and the internal field, the final equation is

$$\frac{\mu_2}{\mu_1} = \frac{\omega\gamma K(1 + NK)H_a}{\gamma^2 H_a^2(1 + K) - \omega^2(1 + NK)^2}. \tag{4}$$

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An Improved Method for the Determination of Q of Cavity Resonators*

AMARJIT SINGH†

Summary—**The various Q factors and circuit efficiency of a cavity resonator can be evaluated from standing-wave measurements on a transmission line or waveguide coupled to the resonator. In the usual method, measurement errors near the half-power points have an unduly large influence on the result. This paper describes a method in which this type of error is avoided.**

In the new method, vswr and position of minimum at various frequencies are plotted on a Smith chart and a circle is drawn through the points. This circle is suitably rotated around the center of the chart and a value of equivalent susceptance is read off for each frequency. The graph of susceptance vs frequency is a straight line, from whose slope the Q factors are evaluated.

The underlying theory of the above method is discussed and typical experimental results are presented. Charts of parameters required in the calculations are given.

Introduction

THE measurement of the Q of cavity resonators finds many applications in the field of microwave electronics, as well as in physical research. Several methods of determination of Q have been developed.[1–3] Among these is the method involving measurements on standing waves in a transmission line or waveguide coupled to the resonator. The method is not a quick one; however, it has the advantage of supplying the most complete information about the resonator and the coupling system. The losses inside the cavity, the losses in the coupling system, and the power coupled into the transmission line, can all be separated out.[4] This information is indispensable in such applications as design of microwave tubes, where the circuit efficiency is an important parameter.

After data have been obtained on the variation of vswr and the position of voltage minimum in the line as functions of frequency, it is possible to determine the frequencies which correspond to the half-power points. The Q factors can then be evaluated. The previously developed methods of obtaining the half-power frequencies employ a curve showing vswr vs frequency or one showing the position of the minimum vs frequency. On these curves, one reads off the frequencies which correspond to certain values of vswr or of shift in the position of the minimum. It is to be noted that these curves are not geometrically simple ones, and that any errors in the observations near the half-power points have a large influence on the result. In fact, a discrepancy may be observed between the results obtained

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[1] C. G. Montgomery, "Technique of Microwave Measurements," McGraw-Hill Book Co., Inc., New York, N. Y.; 1947.

[2] H. M. Barlow and A. L. Cullen, "Microwave Measurements," Constable and Co., London, Eng.; 1950.

[3] M. Wind and H. Rapaport, "Handbook of Microwave Measurements," Polytechnic Institute of Brooklyn, Brooklyn, N. Y., vols. I and II; 1955.

[4] L. Malter and G. R. Brewer, "Microwave Q measurements in the presence of series losses," *J. Appl. Phys.*, vol. 20, pp. 918–925; October, 1949.

from the vswr curve and the position of minimum curve, based on the same set of observations. It was a rather large discrepancy of this kind which led to the formulation of the method described here.

In the new method, the data yield one set of points which lie on a circle and another set which lie practically on a straight line. Three parameters relating to the circle and the line are used in subsequent calculations. The accuracy of these parameters is thus governed by the readings taken at all frequencies. Also, the observations on vswr as well as position of minimum are utilized simultaneously in the determination of Q. The method has been worked out to take losses in the coupling system into account.

The procedure is illustrated by examples and certain computed parameters are presented in charts. A study relating to the linearity of the curve which is assumed to be a straight line is contained in the Appendix.

Equivalent Circuit and Definitions

It has been shown[5] that a cavity resonator and its coupling system can be represented at frequencies in the vicinity of a resonant mode by an equivalent circuit given in Fig. 1(a). Here the series resistance represents losses in the transmission line and coupling system. The series reactance can be made zero by a suitable choice of the reference plane in the transmission line. With such a choice, the equivalent circuit in the case of a matched line coupled to the resonator is given by Fig. 1(b).

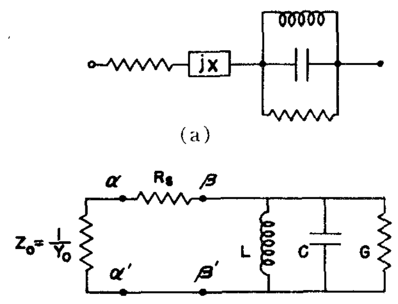


Fig. 1—(a) Equivalent circuit for a cavity resonator near a mode of resonance. (b) Equivalent circuit for a cavity resonator coupled to a matched transmission line, at a reference plane so chosen as to make $X=0$.

Also, it was shown[5] that if the vswr and position of minimum in the transmission line are determined as functions of frequency near resonance and if the corresponding points are plotted on a Smith chart, they lie on a circle, as shown in Fig. 2. A shift in the location of the reference plane on the transmission line is equivalent to a rotation of this circle around the center of the Smith chart. Let the circle be rotated in such a way that its center lies on the radius of the Smith chart passing through the infinity point. The circle then represents either the admittance of an equivalent parallel resonant circuit, or the impedance of an equivalent series resonant circuit. The former representation is used here and the circle is termed the admittance circle.

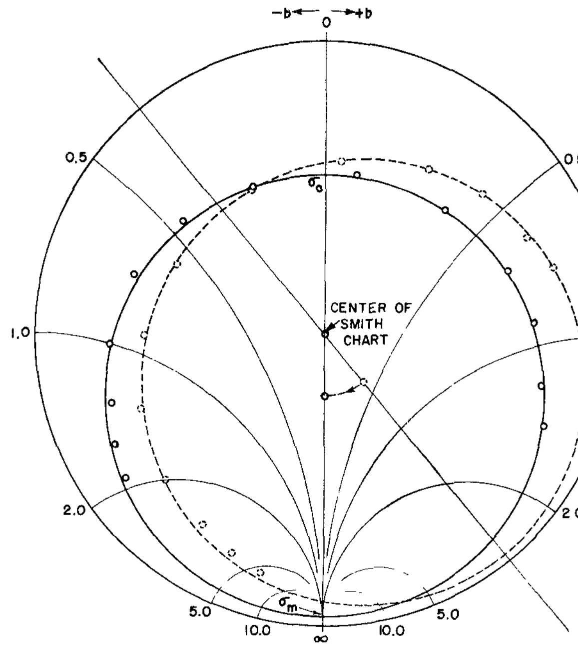


Fig. 2—Admittance circle on Smith chart, as drawn first and after rotation corresponding to suitable choice of reference plane.

At the resonance point, the vswr is a minimum and is denoted by σ_0. At frequencies very far away from resonance the vswr is a maximum and is denoted by σ_m. Two cases can be distinguished,[4] depending upon whether or not the circle just described encloses the center of the Smith chart. When the center is enclosed, the power loss in Z_0 is greater than that in R_s and G. When the center is not enclosed, the power loss in Z_0 is less than that in R_s and G. The two cases are referred to as "overcoupled" and "undercoupled" respectively.

The values of σ_m and σ_0 are related to the parameters of the equivalent circuit as follows:

$$\sigma_m = \frac{Z_0}{R_s} = \frac{1}{r_s}, \tag{1}$$

$$\sigma_0 = \left(R_s + \frac{1}{G}\right) \Big/ Z_0 = r_s + \frac{1}{g}, \tag{2}$$

in the overcoupled case, and

$$1/\sigma_0 = r_s + \frac{1}{g}, \tag{3}$$

in the undercoupled case, where r_s and g are normalized values of R_s and G respectively.

[5] C. G. Montgomery, R. H. Dicke, and E. M. Purcell, "Principles of Microwave Circuits," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 226–227; 1948.

Outline of the Method

It is shown in the Appendix that near resonance the normalized susceptance b_α, appearing to the right at terminals $\alpha\alpha'$, varies practically linearly with frequency. The values of b_α at various frequencies and of σ_0, σ_m can be determined from the admittance circle drawn on a Smith chart. If b_α is plotted as a function of f, an accurate value of df/db_α can be obtained from the slope of the curve.

Next, the Q factor denoted by Q_{00} by Malter and Brewer is evaluated. Q_{00} takes into account only the losses within the resonator proper (represented by G in the equivalent circuit). It can be determined as follows:

$$Q_{00} = \frac{f_0}{f_2 - f_1}, \tag{4}$$

where f_1 and f_2 are the frequencies at which $b_\beta = \pm g$, b_β being the normalized susceptance appearing towards the right at the terminals $\beta\beta'$. Let the value of b_α, when $b_\beta = g$, be denoted by $(b_\alpha)_{00}$. Then Q_{00} is given by

$$Q_{00} = \frac{f_0}{2\dfrac{df}{db_\alpha}(b_\alpha)_{00}}. \tag{5}$$

The other Q factors can be evaluated from a knowledge of Q_{00}, σ_m, σ_0, through the use of a procedure which is the reverse of that described by Malter and Brewer.[4] The circuit efficiency can also be determined in the usual way.

Evaluation of $(b_\alpha)_{00}$

From the equivalent circuit of Fig. 1(b), it can be seen that the normalized admittance y_α appearing to the right at terminals $\alpha\alpha'$ is given by:

$$y_\alpha = \left(r_s + \frac{1}{g + jb_\beta}\right)^{-1}. \tag{6}$$

This gives

$$b_\alpha = \frac{b_\beta}{(1 + r_s g)^2 + r_s^2 b_\beta^2}. \tag{7}$$

The value of b_α, when b_β takes the value g, is given by:

$$(b_\alpha)_{00} = \frac{g}{1 + 2r_s g + 2r_s^2 g^2}. \tag{8}$$

Also, from (1) through (3), we have:

$$g = \begin{cases} \sigma_m/(\sigma_0\sigma_m - 1), & \text{for overcoupled case,} \\ \sigma_m\sigma_0/(\sigma_m - \sigma_0), & \text{for undercoupled case,} \end{cases} \tag{9}$$

and

$$r_s g = \begin{cases} 1/(\sigma_0\sigma_m - 1) & \text{for overcoupled case,} \\ \sigma_0/(\sigma_m - \sigma_0), & \text{for undercoupled case.} \end{cases} \tag{10}$$

Substituting (9) and (10) in (8), we get:

$$(b_\alpha)_{00} = \begin{cases} \dfrac{\sigma_m(\sigma_0\sigma_m - 1)}{(\sigma_m^2\sigma_0^2 + 1)}, & \text{for overcoupled case,} \\[2ex] \dfrac{\sigma_0\sigma_m(\sigma_m - \sigma_0)}{(\sigma_m^2 + \sigma_0^2)}, & \text{for undercoupled case.} \end{cases} \tag{11}$$

In the limiting case of zero series losses, σ_m is infinity and $(b_\alpha)_{00}$ is equal to $1/\sigma_0$ for the overcoupled case and to σ_0 for the undercoupled case. For finite σ_m, one may put

$$(b_\alpha)_{00} = \begin{cases} \dfrac{F_0}{\sigma_0}, & \text{for overcoupled case,} \\ F_u\sigma_0, & \text{for undercoupled case,} \end{cases} \tag{12}$$

where

$$F_0 = \frac{\sigma_0\sigma_m(\sigma_0\sigma_m - 1)}{(\sigma_m^2\sigma_0^2 + 1)}, \tag{13}$$

$$F_u = \frac{\sigma_m(\sigma_m - \sigma_0)}{\sigma_m^2 + \sigma_0^2}. \tag{14}$$

The values of the factors F_0 and F_u have been computed for various combinations of the values of σ_m and σ_0, and are given in Fig. 3(a) and 3(b) (next page).

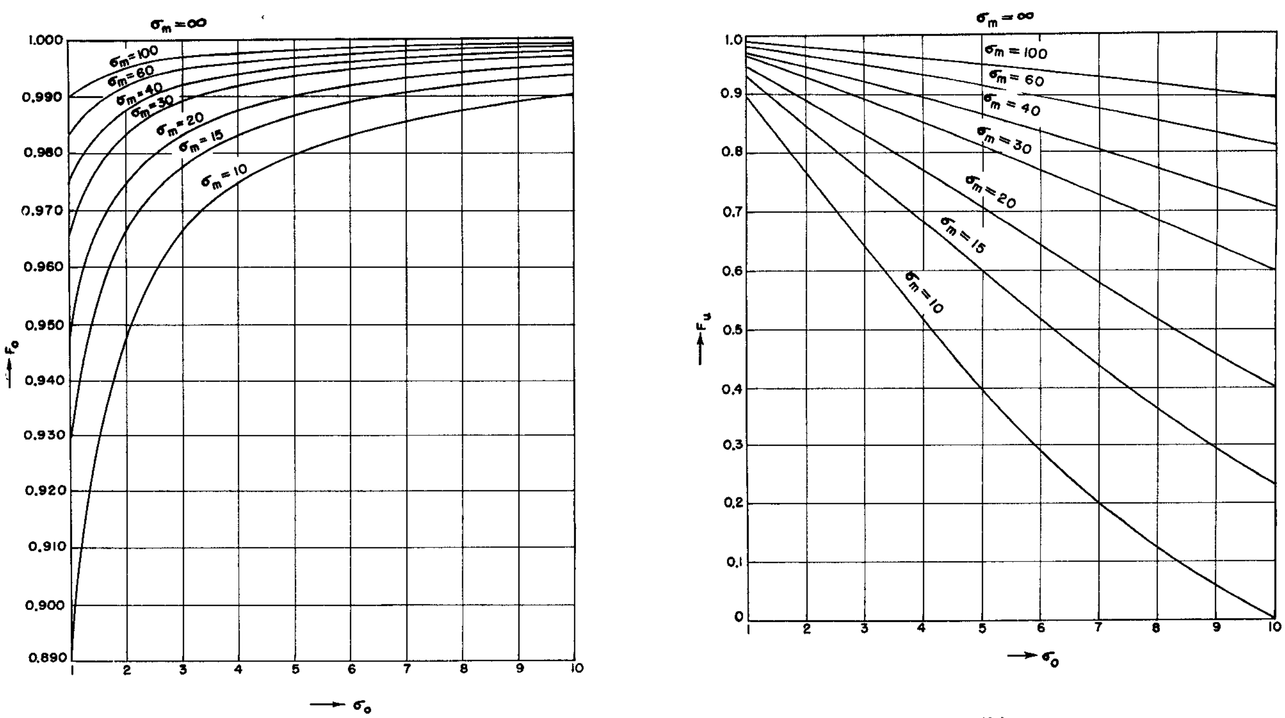
Experimental Procedure

Observations are first taken on the variation of vswr and the position of the minimum with frequency in the vicinity of resonance. The position of the minimum on the transmission line at a frequency near resonance is arbitrarily chosen as a reference point. The shift of the minimum with respect to this point is determined and is corrected for dispersion in the portion of transmission line between the cavity and the probe.[6] The corrected shift is expressed in terms of wavelength. With these data, points corresponding to various frequencies are plotted on a Smith chart. A circle is drawn through these points. When the best circle is drawn through the points, any large errors in standing-wave measurements tend to be smoothed out. The points and the circle are then swung around the center of the Smith chart, so as to bring the center of the admittance circle on that radius of the Smith chart which passes through the infinity point.

The above mentioned procedure involving rotation of the admittance circle can be conveniently performed if a piece of tracing paper is attached to the Smith chart. The points, the circle, and the center of the Smith chart are plotted on the tracing paper, which can then be easily rotated about the center of the Smith chart.

The values of b_α corresponding to each frequency, and

[6] *Ibid.*, p. 339.



(a) (b)

Fig. 3—Functions to evaluate normalized susceptance as seen from terminals $\alpha\alpha'$, when that seen from $\beta\beta'$ is equal to g:

(a) for overcoupled case, $(b_\alpha)_{00} = F_0/\sigma_0$, where

$$F_0 = \frac{\sigma_0\sigma_m(\sigma_0\sigma_m - 1)}{(\sigma_m^2\sigma_0^2 + 1)},$$

(b) for undercoupled case, $(b_\alpha)_{00} = F_u\sigma_0$, where

$$F_u = \frac{\sigma_m(\sigma_m - \sigma_0)}{\sigma_m^2 + \sigma_0^2}.$$

also the values of σ_0, σ_m, can now be read off from the Smith chart.

The values of b_α are plotted as a function of frequency. In drawing a straight line through them, any large errors in frequency determination also tend to be smoothed out. The slope of the line gives db_α/df and the value of f at $b_\alpha = 0$, gives f_0.

After it is decided whether the case is one of overcoupling or undercoupling, as explained earlier, the value of $(b_\alpha)_{00}$ is determined from (12) and Fig. 3(a) and 3(b). Q_{00} is obtained by a substitution of the values found for df/db_α, f_0, and $(b_\alpha)_{00}$ in (5).

Q_0 and Q_L can then be derived from the following relations given by Malter and Brewer.[4]

$$Q_0 = \begin{cases} Q_{00}/D_0, & \text{for overcoupled case,} \\ Q_{00}/D_u, & \text{for undercoupled case,} \end{cases} \tag{15}$$

where

$$D_0 = 1 + \frac{\sigma_0\sigma_m - 1}{(1 + \sigma_m)^2}, \tag{16}$$

and

$$D_u = 1 + \frac{\sigma_m - \sigma_0}{\sigma_0(1 + \sigma_m)^2}. \tag{17}$$

Also,

$$Q_L = \begin{cases} Q_0/(1 + \sigma_0)H_0, & \text{for overcoupled case,} \\ Q_0/(1 + 1/\sigma_0)H_u, & \text{for undercoupled case,} \end{cases} \tag{18}$$

where

$$H_0 = \frac{\sigma_m + 1}{\sigma_m + \sigma_0 + 2}, \tag{19}$$

and

$$H_u = \frac{\sigma_{m+1}}{\sigma_m + \dfrac{1}{\sigma_0} + 2}. \tag{20}$$

Examples

The procedure just described has been tried out in a large number of cases in S band and also in the uhf region. The admittance circle and the b_α vs f curve obtained in a typical case, in the uhf region, are shown in Figs. 2 and 4, respectively. Fig. 2 shows the admittance circle as plotted in its initial position and also after rotation. Fig. 4 shows the b_α vs f curve which is nearly a straight line over a range of frequencies about six times $(f_1 - f_2)$. Relevant parameters for this case are tabulated in Table I.

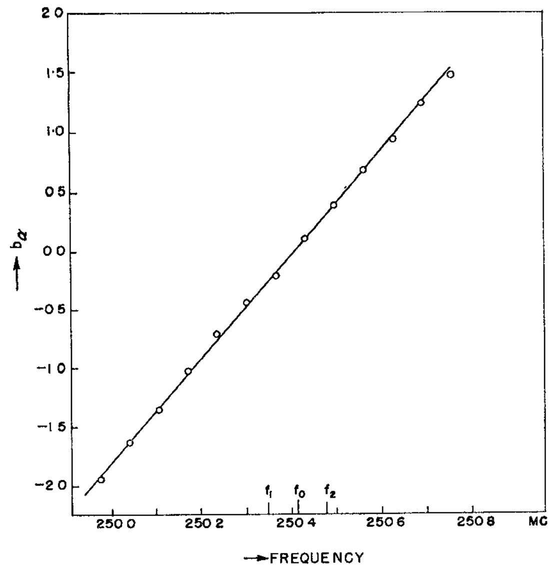


Fig. 4—Plot of normalized susceptance seen at reference plane as a function of frequency.

than 2 per cent between the half-power frequencies, for $Q_L=25$, $\sigma_0=1$, and $\sigma_m=20$. Under normal conditions the values of Q_L, σ_m, and coupling are greater than those assumed above. Thus b_α varies linearly with f over a frequency range substantially larger than the separation between half-power frequencies.

All the data over this extended frequency range contribute to the accuracy of the final result. The accuracy is somewhat reduced by the fact that error can be introduced when rotating the admittance circle on the Smith chart and in reading off susceptance values. However, it can be seen from the results presented in Table II and Table III that there is an over-all decrease in the standard deviation, when the new method is used.

The difference between the mean values obtained by using the two methods needs further investigation. Errors in the absolute values of vswr can cause such a difference. It is also possible that the difference arises

TABLE I

f_0	σ_m	σ_0	F_0	$(b_\alpha)_{00}$	df/db_α	Q_{00}	Q_0	Q_L	Circuit efficiency
250.4 mc	100	3.45	0.996	0.288	0.226 mc	1931	1875	434	76.8 per cent

TABLE II

Values of Q_L for the Individual Sets of Readings

Set no.	Q_L as determined by the new method	Q_L as determined by the method of Malter and Brewer
1	999	975
2	978	1045
3	1010	1065
4	1020	1112
5	977	976
6	1050	1060
7	1012	1155
8	990	1011
9	996	1125
10	1025	1047

TABLE III

Values of Parameters Deduced from All the Sets

Parameter	Value from new method	Value from Malter and Brewer's method
f_0	2.892 kmc	
σ_0 (overcoupled case)	2.46 to 2.57	
σ_m	100 or above	
Mean Q_L	1006	1057
Standard deviation of Q_L	21	58
Mean Q_0	3470	3640
Standard deviation of Q_0	80	190
Mean Q_{00}	3520	3700
Standard deviation of Q_{00}	90	210

In order to determine the accuracy obtained by using the new method, as compared with that obtained by using the method given by Malter and Brewer,[4] ten sets of observations were taken on a cavity with a resonant mode in the S band. The values of Q_L obtained from the individual sets of data, using the two methods, are tabulated in Table II. The values of parameters deduced from all the sets are given in Table III. It can be seen that the standard deviation (also referred to as root mean square deviation) obtained by the new method is less than one half of that obtained by the half-power point method.

Discussion of Accuracy

An analysis given in the Appendix shows that the variation of b_α with f deviates from linearity by less

because the simplified equivalent circuit used as a common starting point for the two methods is not an exact representation of the cavity resonator.

Conclusion

A method of evaluating the Q of cavity resonators from vswr data has been presented. The accuracy of the final result does not depend unduly on readings taken close to the half-power points, but upon all the readings of vswr as well as position of minimum. Any large errors are smoothed out at two stages in the procedure. This procedure has been worked out to cover the case in which losses in the transmission line and the coupling system are taken into account, and these losses can be isolated from other losses. The method is particularly suitable for applications in which accurate values of Q_L, Q_0, Q_{00}, Q_{ext}, and the circuit efficiency are required.

Appendix

Linearity of Variation of b_α With Frequency

Let us first consider the variation of b_β with frequency

$$b_\beta = \frac{1}{Y_0}\left(\omega C - \frac{1}{\omega L}\right)$$

$$= \frac{1}{Y_0}\sqrt{\frac{C}{L}}\,\frac{2\delta}{f_0}\left(1 + \frac{\delta}{2f_0}\right)\left(1 + \frac{\delta}{f_0}\right)^{-1},$$

where

$$\delta = f - f_0.$$

or

$$b_\beta = \frac{1}{Y_0}\sqrt{\frac{C}{L}}\,\frac{2\delta}{f_0}\left(1 - \frac{\delta}{2f_0} + \text{higher order terms in } \frac{\delta}{f_0}\right)$$

$$= \frac{1}{Y_0}\sqrt{\frac{C}{L}}\,\frac{2\delta}{f_0}\left(1 - \frac{\delta}{\delta_L}\,\frac{1}{4Q_L} + \cdots\right), \tag{21}$$

where $2\delta_L = f_0/Q_L$ = difference between half-power frequencies. Also, from (7) we have

$$b_\alpha = \frac{b_\beta}{(1 + r_s g)^2 + r_s^2 b_\beta^2}$$

$$= \frac{b_\beta}{(1 + r_s g)^2}\left[1 - \left(\frac{r_s b_\beta}{1 + r_s g}\right)^2 + \cdots\right].$$

In the second term in the square brackets, a first-order approximation for b_β can be used since the second term will be small in comparison with the first term. As shown by (21), b_β is proportional to δ, to a first approximation, and also $b_\beta \cong (1+g)$, for $\delta = \delta_L$. Thus to a first approximation, $b_\beta = (1+g)\delta/\delta_L$.

Making this substitution in the second term in the square brackets, we get:

$$b_\alpha = \frac{b_\beta}{(1 + r_s g)^2}\left[1 - \left(\frac{r_s}{1 + r_s g}\right)^2 (1 + g)^2 \left(\frac{\delta}{\delta_L}\right)^2 + \cdots\right].$$

Substituting for r_s and g from (1) and (9), and for the remaining factor b_β from (21), and neglecting[7] higher order terms in $1/\sigma_m$, we get:

$$b_\alpha = K_0\delta\left(1 - \frac{\delta}{\delta_L}\,\frac{1}{4Q_L} + \cdots\right) \cdot\left[1 - \left(\frac{\sigma_0 + 1}{\sigma_0\sigma_m}\right)^2\left(\frac{\delta}{\delta_L}\right)^2 + \cdots\right], \tag{22}$$

for the overcoupled case, and

$$b_\alpha = K_u\delta\left(1 - \frac{\delta}{\delta_L}\,\frac{1}{4Q_L} + \cdots\right) \cdot\left[1 - \left(\frac{1 + \sigma_0}{\sigma_m}\right)^2\left(\frac{\delta}{\delta_L}\right)^2 + \cdots\right], \tag{23}$$

for the undercoupled case, where,

$$K_0 = \frac{1}{Y_0}\sqrt{\frac{C}{L}}\,\frac{2}{f_0}\left(\frac{\sigma_0\sigma_m - 1}{\sigma_0\sigma_m}\right)^2, \tag{24}$$

and

$$K_u = \frac{1}{Y_0}\sqrt{\frac{C}{L}}\,\frac{2}{f_0}\left(\frac{\sigma_m - \sigma_0}{\sigma_m}\right)^2. \tag{25}$$

As K_0 and K_u do not depend upon f, it can be seen that b_α is proportional to δ to a first approximation. The correction terms have been expressed as functions of the ratio of $(f - f_0)$ to the difference between half-power frequencies, δ/δ_L. It is also seen that the corrections are smaller for larger values of Q_L, for smaller series losses (*i.e.*, larger σ_m), and for greater coupling between cavity and line (*i.e.*, larger σ_0 for overcoupled case and smaller σ_0 for undercoupled case).

In particular, if $Q_L = 25$, then the correction due to the first factor is less than 1 per cent over a range of frequencies lying between the half-power points. Similarly if $\sigma_m = 20$, then at critical coupling, $\sigma_0 = 1$, the correction due to the second factor is less than 1 per cent over the same range. The deviations from linearity would be largest when coupling is small and at the same time the series losses are appreciable.

Thus for most practical purposes, the variations of b_α with frequency may be considered linear to a good approximation, over frequency ranges of interest.

Acknowledgment

The author is grateful to K. S. Krishnan and K. N. Mathur for their kind encouragement. He is also indebted to N. C. Vaidya, M. S. Kunchitham, and R. A. Rao who used the method in a large number of cases and thus helped in its verification.

[7] The exact expressions for coefficients of $(\delta/\delta_L)^2$ are

$$[(\sigma_0\sigma_m + \sigma_m^{-1})/(\sigma_0\sigma_m^2)]^2 \quad\text{and}\quad [(\sigma_m + \sigma_m\sigma_0 - \sigma_0)/\sigma_m^2]^2$$

for the overcoupled and undercoupled cases, respectively. The simpler expressions mentioned above give good approximations since σ_m is normally $\gg 1$, and since the coefficients in question are themselves correction terms.